MTH 406: Differential geometry of curves and surfaces

Homework VI: Surface area, First fundamental form, and Conformal maps

Problems for practice

1. Establish the assertions in 2.6(iv) , 2.7(ii), and 2.7(iv)-(vi) of the Lesson Plan.

2. Derive a general formula for the area of a surface of revolution in 2.2(viii)(a) of the Lesson Plan. Use this to compute the surface area of the torus T^2 in 2.1(ii)(b) of the Lesson Plan, as a surface of revolution.

3. Compute the area of the Möbius band M in 2.4(vii)(e) of the Lesson Plan.

4. Let $f : S_1 \to S_2$ be a diffeomorphism between regular surfaces. Prove that f is an isometry if, and only if, for every curve $\gamma : [a, b] \to S_1$, the length of γ equals the length of $f \circ \gamma$.

5. A function $f : S_1 \to S_2$ is a *local diffeomorphism (resp. isometry)* if for each $p \in S_1$, there exists a neighborhood $U_p \ni p$ such that $f|_{U_p} : U_p \to f(U_p)$ is a diffeomorphism (resp. isometry).

   (a) Show that the map $f : \mathbb{R}^2 \to C : (x, y) \overset{f}{\mapsto} (\cos(x), \sin(x), y)$, where
   $$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$
   is a local isometry. Show that these surfaces are not globally diffeomorphic.

   (b) Construct a local diffeomorphism from the cylinder
   $$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z \in (-1/2, 1/2)\}$$
   to the Möbius strip. Show that these surfaces are not globally diffeomorphic.

6. Let $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \overset{f}{\mapsto} xy$. Classify the rigid motions of $\mathbb{R}^3$ that induce isometries on G_f.

7. Let $f : S \to \tilde{S}$ be a diffeomorphism of regular surfaces. Show that the following statements are equivalent.

   (a) f is an isometry.

   (b) For every coordinate patch $\sigma : U \to V(\subset S)$, the first fundamental form of σ equals the first fundamental form of $f \circ \sigma$.

   (c) Every $p \in S$ is covered by a coordinate patch $\sigma : U \to V(\subset S)$ such that the first fundamental form of σ equals the first fundamental form of $f \circ \sigma$.

8. Compute the first fundamental forms for the sphere S^2 (determined by the stereographic projection), the cylinder, the torus T^2, and the Möbius band M.

9. Let S be a regular surface, and $f : U \to V(\subset S)$ is a surface patch with the first fundamental form $Efdu^2 + 2Fdu\,dv + Gdv^2$. Show that:

   (a) f is an isometry if, and only if, $E = G = 1$, and $F = 0$.

   (b) f is an equiareal map if, and only if, $\sqrt{EG - F^2} = 1$.

   (c) f is a conformal map if, and only if, $E = G$, and $F = 0$.